The Onset of Congestion in Charging of Electric Vehicles for Proportionally Fair Network Management Protocol

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1 Introduction

With the expected uptake of electric vehicles in the near future, we are likely to observe overloading in the local distribution networks more frequently. Such development suggests that a congestion management protocol will be a crucial component of future technological innovations in low voltage networks. An important property of a suitable network capacity management protocol is to balance network efficiency and fairness requirements. Assuming a stochastic model, we study the proportional fairness (PF) protocol managing the network capacity in charging of electric vehicles. We explore the onset of congestion by analysing the critical arrival rate, i.e. the largest possible vehicle arrival rate that can still be fully satisfied by the network. We compare the proportionally fair management protocol with the max-flow (MF) management protocol. By numerical simulations on realistic networks, we show that proportional fairness leads not only to more equitable distribution of power allocations, but it can also serve slightly larger arrival rate of vehicles. We consider simplified setup, where the power allocations are dependent on the occupation of network nodes, but they are independent of the exact number of vehicles, and to validate numerical results, we analyse the critical arrival rate on a network with two edges, where the optimal power allocations can be calculated analytically.

2 Optimization Model

We model the electrical distribution network as a directed rooted tree graph composed of the node set $\mathcal{V}$ and edge set $\mathcal{E}$. Only the root node of the tree $r \in \mathcal{V}$ injects the power into the network and electric vehicles can be plugged into all other nodes. By

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the symbol $\pitchfork(j)$ we denote the subtree rooted in the node $j \in \mathcal{V}$. An edge $e_{ij} \in \mathcal{E}$ connects node i to node j, where i is closer to the root than j, and is characterised by the impedance $Z_{ij} = R_{ij} + i X_{ij}$, where R_{ij} is the edge resistance and X_{ij} the edge reactance. The power loss along edge e_{ij} is given by $S_{ij}(t) = P_{ij}(t) + i Q_{ij}(t)$, where $P_{ij}(t)$ is the real power loss, and $Q_{ij}(t)$ the reactive power loss. We model car batteries as elastic loads (i.e. able to absorb any value of power they are allocated). Electric vehicle $l = 1, \ldots, N(t)$ receives only active power $P_l(t)$, where $N(t)$ is the number of vehicles charging at time t. Value $\Delta_{il}(t)$ is one if electric vehicle l is charging on node i and zero otherwise. Vehicle l derives a utility $U_l(P_l(t))$ from the allocated charging power $P_l(t)$. Let $P_{\pitchfork(j)}$ denote the active power, and $Q_{\pitchfork(j)}$ the reactive power consumed by the subtree $\pitchfork(j)$ that include power consumed by all vehicles connected to the subtree and power losses dissipated on edges of the subtree. By the symbol $V_i(t)$, we denote the voltage level on the node $i \in \mathcal{V}$. We allocate the power to electric vehicles by maximizing the aggregate utility $U(t)$, while making sure that all nodal voltages are within the interval $((1-\alpha)V_{\text{nominal}}, (1+\alpha)V_{\text{nominal}})$, where α is a parameter and V_{nominal} is the nominal voltage level the network is operated on (for more details see Ref. [1]):

$$\underset{W(t)}{\text{maximise}} \quad U(t) = \sum_{l=1}^{N(t)} U_l(P_l(t)) \tag{1}$$

$$\text{subject to} \quad ((1-\alpha)V_{\text{nominal}})^2 \leq W_{ii}(t) \leq ((1+\alpha)V_{\text{nominal}})^2, \qquad i \in \mathcal{V}, \tag{2}$$

$$W_{ij}(t) - W_{jj}(t) - P_{\pitchfork(j)}(t)R_{ij} - Q_{\pitchfork(j)}(t)X_{ij} = 0, \qquad e_{ij} \in \mathcal{E}, \tag{3}$$

$$\begin{pmatrix} W_{ii}(t) & W_{ij}(t) \\ W_{ji}(t) & W_{jj}(t) \end{pmatrix} \succeq 0, \qquad e_{ij} \in \mathcal{E}. \tag{4}$$

With every edge $e_{ij} \in \mathcal{E}$ is associated one decision variable $W_{ij}(t)$ that it is equal to the product of real voltages on edge nodes, i.e. $W_{ij}(t) = V_i(t)V_j(t)$ and similarly with every node $i \in \mathcal{V}$ is associated variable $W_{ii}(t) = V_i(t)^2$. The generalized inequality (4) means that matrices are positive semidefinite. Constraints (2) ensure that all nodal voltages are within the defined limits. Constraints (3)–(4) have been derived in reference [1] and they encode relations between decision variables $W_{ij}(t)$, power allocations $P_l(t)$ and power losses along edges that arise from Kirchhoff's current and voltage laws, where:

$$P_{\pitchfork(k)} = \sum_{i \in \mathcal{V}_{\pitchfork(k)}} \sum_{l=1}^{N(t)} \Delta_{il}(t) P_l(t) + \sum_{i \in \mathcal{V}_{\pitchfork(k)}} \sum_{j: e_{ij} \in \mathcal{E}_{\pitchfork(k)}} P_{ij}(t), \tag{5}$$

and

$$Q_{\pitchfork(k)} = \sum_{i \in \mathcal{V}_{\pitchfork(k)}} \sum_{j: e_{ij} \in \mathcal{E}_{\pitchfork(k)}} Q_{ij}(t), \tag{6}$$

where power losses along edge $e_{ij} \in \mathcal{E}$ can be expressed as:

$$P_{ij}(t) = \big(W_{ii}(t) - 2W_{ij}(t) + W_{jj}(t)\big)\, \frac{R_{ij}}{R_{ij}^2 + X_{ij}^2}, \tag{7}$$

and

$$Q_{ij}(t) = \big(W_{ii}(t) - 2W_{ij}(t) + W_{jj}(t)\big)\, \frac{X_{ij}}{R_{ij}^2 + X_{ij}^2}. \tag{8}$$

We consider the *proportional fair* allocation representing a trade-off between network throughput and equality in allocations [2], that maximizes the sum of the logarithm of user rates, i.e. $U(t) = \sum_{l=1}^{N(t)} \log(P_l(t))$. Computationally it is more practical to use the equivalent definition $U(t) = \sum_{i\in\mathcal{V}^+} w_i(t) \log(\mathrm{P_i}(t))$, where $\mathrm{P_i(t)}$ is power allocated to network node i, $\mathcal{V}^+$ is the subset of nodes with at least one charging vehicle, and $w_i(t)$ is a number of vehicles charging at node i at time t, i.e. $w_i(t) = \sum_{l=1}^{N(t)} \Delta_{il}(t)$. Values $P_l(t)$ can be then recovered from $P_l(t) = \frac{\mathrm{P_i}(t)}{w_i(t)}$. As a benchmark representing the efficient network throughput, we consider non-unique *max-flow* allocation given by $U(t) = \sum_{l=1}^{N(t)} P_l(t)$, where we optimise the system whenever the configuration of vehicles changes. Max-flow maximizes the network throughput, however, it can leave some users with zero power, which can be considered as unfair from the user point of view. Both problems are convex, and hence can be solved by general purpose optimization solvers.

To study the behaviour of proportional fairness and max-flow, we implemented a discrete simulator that solves the problem (1)–(4) in discrete time steps. Simulations start with empty network. Vehicles arrive to the network in continuous time, following a Poisson process with rate λ, and choose node to charge randomly with uniform probability. Vehicles have a battery with capacity B that is empty at arrival and leave the network when it is full. The level of battery is given by the time integral of allocated power.

3 Results

3.1 Numerical Experiments

We simulate vehicles charging on the realistic SCE 47-bus network [3] while setting $V_{\mathrm{nominal}} = B = 1.0$ and $\alpha = 0.1$. In order to characterize the behaviour of the network, we adopt the congestion parameter [4]:

$$\eta(\lambda) = \lim_{t\to\infty} \frac{1}{\lambda} \frac{\langle \Delta N(t)\rangle}{\Delta t}, \tag{9}$$

where $\Delta N(t) = N(t+\Delta t) - N(t)$ and $\langle \dots \rangle$ indicates an average over time window of length Δt. Congestion parameter $\eta(\lambda) = 0$ when all cars leave the network fully charged within a large enough time window, and $\eta(\lambda) > 0$, when some vehicles

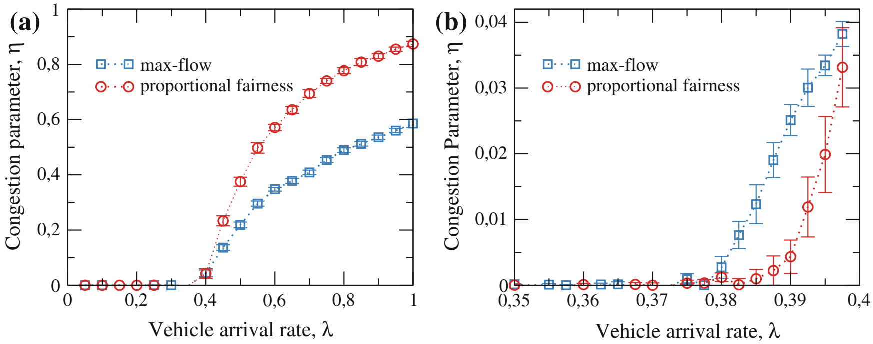

Fig. 1 **a** Congestion parameter η as a function of the vehicle arrival rate for the SCE 47-bus network and for the simulation time horizon of 1.5×10^4. **b** Zoom of the critical region for longer horizon of 10^5 time units. Symbols show average values over an ensemble of 25 independent runs and error bars reflect 95 % confidence intervals

have to wait for increasingly long times to fully charge, i.e. the network is congested. Simulation results in Fig. 1 show that the largest value of the arrival rate λ_c, when all vehicles are still fully charged, is larger for proportional fairness than for max-flow, meaning that proportional fairness is able to charge slightly larger number of vehicles.

3.2 *Onset of Congestion in 2-Edge Network*

To validate analytically that λ_c can be different in both methods, we analyse a three-node and two-edge network with node 1 (root node), and vehicles arriving at node 2 (the closest node to the root), and at node 3 (the leaf node), respectively, assuming uniform R and X values.

The two congestion control methods lead to different allocations of instantaneous power, with vehicles charging in different order and in different time intervals. The voltage drops with the increasing distance from the root and the lower voltage limit (constraint (2)) is fulfilled at equality for one node. The objective function of proportional fairness guaranties that both nodes (if occupied by vehicles) will receive positive power allocation. Thus, the lower voltage limit constraint is satisfied at equality on the most distant node from the root. In max-flow, however, the maximisation of the aggregate power allocated to vehicles implies also minimising instantaneous power losses, and this is achieved by allocating all power to the closest occupied node from the root node.

Note that optimal max-flow allocation is independent of how many vehicles are charging on each node. To simplify our analysis, we set w_i to value one if at least one vehicle is charging at node i, and zero otherwise for $i \in 2, 3$, and thus proportional fair optimal power allocations will be also independent of the number of vehicles on each node. For this simplified setup we can easily estimate the critical value λ_c analytically.

Under our assumptions, for 2-edge network the problem (1)–(4) can be solved analytically. Optimal power allocation of max-flow at the node $i \in \{2, 3\}$ is:

$$P_i^{MF} = \frac{2\alpha(1-\alpha)V_{\text{nominal}}^2}{(i-1)R}. \tag{10}$$

Optimal proportional fair power allocations are:

$$P_2^{PF} = \frac{2V_{\text{nominal}}^2(3\sqrt{\gamma}-\gamma)}{9R} \text{ and } P_3^{PF} = \frac{(1-\alpha)V_{\text{nominal}}^2(\sqrt{\gamma}+3\alpha-3)}{3R}, \tag{11}$$

where $\gamma = 2\sqrt{\alpha^2-\alpha+1}|\alpha-2|+2\alpha^2-5\alpha+5$. When deriving the value λ_c, we assume a time interval Δt that is composed of two subintervals t_i, when vehicles situated at node $i \in \{2, 3\}$ are charged. Within this time interval the demand arriving at each of the nodes (i.e. $\frac{B\lambda_c\Delta t}{2}$) has to be the same as the energies $P_2^{MF}t_2$ and $P_3^{MF}t_3$ that max-flow is able to deliver at nodes 2 and 3, respectively. From here we obtain:

$$\lambda_c^{MF} = \frac{P_2^{MF}}{\frac{B}{2}(\frac{P_2^{MF}}{P_3^{MF}}+1)}. \tag{12}$$

Similarly, for proportional fairness we obtain:

$$\lambda_c^{PF} = \frac{P_3^{MF}}{\frac{B}{2}(\frac{P_3^{MF}}{P_2^{PF}}-\frac{P_3^{PF}}{P_2^{PF}}+1)}. \tag{13}$$

We set parameters $R = X = B = V_{\text{nominal}} = 1.0$ and $\alpha = 0.1$, yielding theoretical predictions $\lambda_c^{MF} = 0.12$ and $\lambda_c^{PF} \approx 0.1222$. Thus, our analyses show that proportional fairness may support slightly larger arrival rate, giving support to our numerical simulation on realistic electrical networks. To validate our analyses, we simulated max-flow and proportional fair protocols in 2-edge network with two λ values. When $\lambda < \lambda_c$ number of vehicles is oscillating, while for $\lambda > \lambda_c$ it has a tendency to grow (see Fig. 2). Thus, numerical results are in good agreement with calculated values.

4 Conclusions

The main contribution of this paper is that we showed analytically that PF can accommodate larger arrival rate than MF. This result is surprising, because common expectation is that efficiency of the system comes at the expense of the increased inequality [5]. However, it should be noted that here we optimise the dynamic system over

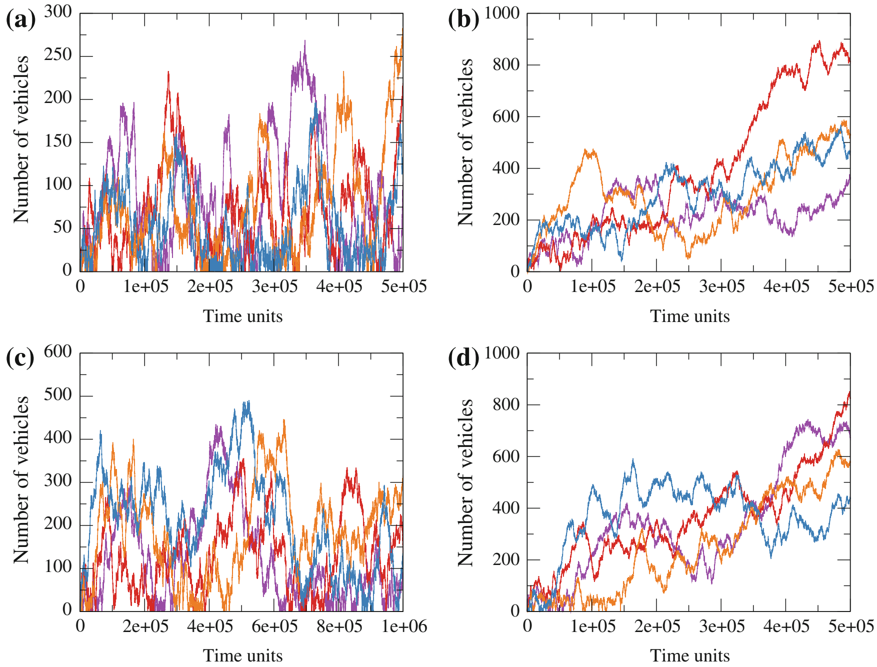

Fig. 2 Representative time series for the 2-edge network. Panel **a** shows that for $\lambda = 0.119$ the max-flow supplies all vehicles, whereas in panel **b**, for $\lambda = 0.121$, it is congested being in the agreement with the calculated value $\lambda_c^{MF} = 0.12$. Panel **c** shows that for $\lambda = 0.122$ the proportional fairness is supplying all vehicles, whereas in panel **d**, for $\lambda = 0.123$, it is congested being in the agreement with the calculated value $\lambda_c^{PF} \approx 0.1222$

a certain time period and our optimisation model is not dynamic, hence, it is only a heuristic.

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